Compare the numbers

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Prove that

lg 2017 · lg 2019 · lg 2020 > lg 2016 · lg 2018 · lg 2022.

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$\log 2017 \cdot \log 2019 \cdot \log 2020 > \log 2016 \cdot \log 2018 \cdot \log 2022 \Leftrightarrow$

$\log_{2016} 2017 \cdot \log_{2018} 2019 > \log_{2020} 2022 \Leftrightarrow$

$\left(1 + \log_{2016}\left(1 + \frac{1}{2016}\right)\right)\left(1 + \log_{2018}\left(1 + \frac{1}{2018}\right)\right) > 1 + \log_{2020}\left(1 + \frac{2}{2020}\right)$.

Noting that $\left(1 + \log_{2016}\left(1 + \frac{1}{2016}\right)\right)\left(1 + \log_{2018}\left(1 + \frac{1}{2018}\right)\right) >$

$1 + \log_{2016}\left(1 + \frac{1}{2016}\right) + \log_{2018}\left(1 + \frac{1}{2018}\right) > 1 + \log_{2020}\left(1 + \frac{1}{2016}\right) + \log_{2020}\left(1 + \frac{1}{2018}\right) =$

$1 + \log_{2020}\left(1 + \frac{1}{2016}\right)\left(1 + \frac{1}{2018}\right) > 1 + \log_{2020}\left(1 + \frac{1}{2016} + \frac{1}{2018}\right)$

and $\frac{1}{2016} + \frac{1}{2018} \geq \frac{4}{2016 + 2018} = \frac{2}{2017} > \frac{2}{2020}$ we complete the proof.

Generalization.

We will consider inequality of the problem in generic terms, namely we will prove that for any $a > 1$ holds inequality

(**1**) $\log(a+1)\log(a+3)\log(a+4) > \log a \log(a+2)\log(a+6) \Leftrightarrow$

$\log_a(a+1) \cdot \log_{a+2}(a+3) > \log_{a+4}(a+6)$.

(by replacing in (**1**) a with 2016 we obtain inequality of the problem).

Since $\log_a(a+1) \cdot \log_{a+2}(a+3) = \left(1 + \log_a\left(1 + \frac{1}{a}\right)\right)\left(1 + \log_{a+2}\left(1 + \frac{1}{a+2}\right)\right) >$

$1 + \log_a\left(1 + \frac{1}{a}\right) + \log_{a+2}\left(1 + \frac{1}{a+2}\right) > 1 + \log_{a+2}\left(1 + \frac{1}{a}\right) + \log_{a+2}\left(1 + \frac{1}{a+2}\right) =$

$1 + \log_{a+2}\left(\left(1 + \frac{1}{a}\right) \cdot \left(1 + \frac{1}{a+2}\right)\right) > 1 + \log_{a+2}\left(1 + \frac{1}{a} + \frac{1}{a+2}\right)$ and

$\frac{1}{a} + \frac{1}{a+2} > \frac{4}{a + (a+2)} = \frac{2}{a+1}$ we obtain

$\log_a(a+1) \cdot \log_{a+2}(a+3) > 1 + \log_{a+2}\left(1 + \frac{2}{a+1}\right) >$

$1 + \log_{a+4}\left(1 + \frac{2}{a+1}\right) > 1 + \log_{a+4}\left(1 + \frac{2}{a+4}\right) = \log_{a+4}(a+6)$.

Remark.

And even more precise $1 + \log_{a+2}\left(1 + \frac{2}{a+1}\right) > 1 + \log_{a+2}\left(1 + \frac{2}{a+2}\right) = \log_{a+2}(a+4)$

that is inequality

(**2**) $\log_a(a+1) \cdot \log_{a+2}(a+3) > \log_{a+2}(a+4)$ holds for any $a > 1$.

(since $\log_{a+2}(a+4) > \log_{a+4}(a+6)$ inequality (**2**) implies inequality (1))